ABSTRACT

BLACKLEDGE, MICHAEL ALLAN. Closest Packing of Equal Spheres and Related Problems. (Under the direction of JOHN MONTGOMERY CLARKSON).

The two-dimensional packing problem is discussed, using the concept of the lattice, and the lattice which determines the closest packing of equal circles is presented. Also, closest packing in terms of density is discussed and the density value for the closest regular packing is derived.

The idea of sphere-clouds is introduced and used as an introduction to the closest packing of spheres. Lattice-like arrangements of spheres are considered, and the density of such a packing is determined.

Two proofs, one by John Leech and one by A. H. **Boerdijk,** are presented to show that it is impossible for thirteen "spheres of equal radius to be in contact with a fourteenth sphere of the same radius.

A second related problem is presented, which when goner-alized reduces to the problem of finding the number of figures with (N + I) vertices in N-spaces choosing the vertices **from** given sets of points on given lines passing through a common point, subject to the restriction that no N lines lie in the same (N -I)-space. A solution by the author is presented and

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CLOSEST PACKING OF EQUAL SPHERES

by

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I. INTRODUCTION

The problem of determining closest packing dates back at least as far as the argument in the marketplace of old of whether the customer was getting "good measure" in his purchase. St. Luke the evangelist writes in Chapter VI, Verse 38: "Give, and it shall be given unto you; good measure, pressed down, and shaken together, and running over, shall men give into your bosom. For with the same measure that ye mete withal it shall be measured to you again."

The purpose of this thesis is to determine the closest possible packing of equal spheres, and to investigate some related problems in this area. To accomplish this purpose, first the two-dimensional problem of determining the closest packing of circles in a plane is considered. Then the analogous problem of packing spheres in Euclidean three-space is investigated, using as a transition between the two problems the sphere cloud concept of L. Fejes T6th.

Among the related problems is that of whether or not it is possible for thirteen spheres of equal size to be in contact with a fourteenth sphere. Two proofs are presented to show that such contact is indeed impossible, the first proof using the points of contact to form a network, and the second using the concept of central projection.

Finally, a problem is presented which generalizes to the idea of determining the number of (N + I)-vertex figures in N-space, choosing the vertices from given points on given

intersectin lines, under the restriction that no N lines lie in the same (N - I)-space. The author presents a solution and compares it with a published solution.

2. CLOSEST PACKING OF CIRCLES

We will consider one packing of circles to be closer than another if a (sufficiently large) prescribed region accomodates more circles of the first packing than of the second.

2.1 Circle Packings and Lattices

TO determine the closest packing of circles, Hilbert and Cohn-Vossen (1952) use the idea of lattices. A square lattice is constructed by marking the four corners of a unit square in the plane. We then move the square one unit of length in the direction parallel to one of its sides, and mark the two new points indicated by the corners. We continue in this manner, and imagine the process to be repeated indefinitely, first in the original direction, and then in the opposite direction. Then we proceed in the directions orthogonal to our original directions, and thus cover the entire plane with points (see Figure I). The totality of these points constitutes the square lattice, and it may be noted that instead of using a unit square to generate the lattice, any parallelogram that can be drawn on the lattice such that no lattice points are within its boundaries, and no lattice points lie on its boundaries except for vertices, may be used to generate the square lattice.

Now we will consider a special case of the general "unit lattices," that Is, lattices that can be constructed from an arbitrary parallelogram of unit area in the manner in which

any such lattices, the minimum distance between any two

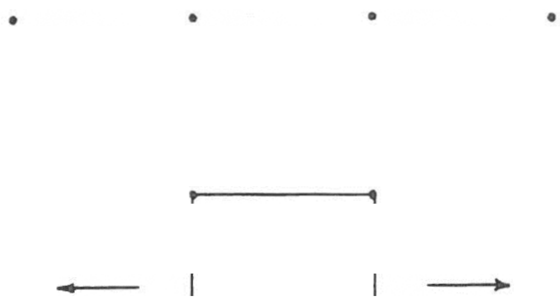

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Figure 1. Construction of the square lattice

$$\frac{1}{d} \geq \frac{d}{2}\sqrt{3}$$

Figure 2. Construction of a unit lattice

$$d \leq \sqrt{\frac{2}{\sqrt{3}}}$$

lattice points is a characteristic quantity. can be arbitrarily small, as in the lattice generated by a rectangle with sides of length and . However, in order for us to be able to generate our lattice in the prescribed manner, must have an upper limit. We shall determine this upper bound.

In any given unit lattice, let us choose any pair of lattice points separated by the minimum distance . By definition of the lattice, there must be infinitely many more points of the lattice lying on the straight line which passes through our two chosen points. By the unit property of the lattice, the straight line parallel to at … distance from must also pass through infinitely many lattice points, yet there are no lattice points between and . We now draw circles of radius with centers at each of the lattice points on (see Figure 2). The area covered by these circles is a strip of the plane bounded by circular arcs. Since every point within this strip is less than the radius from a lattice point, none of these interior points is a lattice point itself, other than the centers. Thus we see that the shortest distance between our line and the boundary of the strip is less than or, in the case where $d = 1$, equal to, . This shortest distance is obviously the altitude of an equilateral

or

and we have found our upper bound for , namely $\sqrt{\frac{2}{\sqrt{3}}}$. Using this idea of the equilateral triangle, we find that there actually is a lattice which attains this maximal value, namely the lattice generated by a parallelogram composed of two equilateral triangles (see Figure 3).

Any lattice of any area can be constructed from a unit lattice simply by increasing or decreasing the dimensions of the unit lattice. Thus if the generating parallelogram has an area of 2 and if D is the minimum dista e between two lattice points, then D is defined by

$$D = a\sqrt{\frac{2}{\sqrt{3}}}$$

Again the maximal distance is achieved if and only if the generating parallelogram is made up of two equilateral triangles.

It has been shown by Hilbert and Cohn Vossen (1952) that the area of large regions is asymptotically equal to the number of lattice points in the region multiplied by the area of the generating parallelogram. Thus for a given minimum distance between lattice points, the lattice built up of equilateral triangles not only has the smallest possible generating parallelogram, it also has the greatest number of points in a given large region.

Now let us construct circles with centers at the lattice points of this equilateral triangle lattice, with the radii of these circles being equal to one-half the minimum distance. Then none of these circles overlap, but tangencies occur. A